Problem 5088. Let a, b be positive integers. Prove that

$$\frac{\varphi(ab)}{\sqrt{\varphi^2\left(a^2\right)+\varphi^2\left(b^2\right)}} \leq \frac{\sqrt{2}}{2}$$

where $\varphi(n)$ is Euler's totient function.

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We will use the following well known result

THEOREM. If n is a natural number then

$$\varphi(n)=\prod_{p|n}\left(1-\frac{1}{p}\right) \tag{1}$$

where $\varphi(n)$ is Euler's totient function and $\prod\limits_{p|n}\left(1-\frac{1}{p}\right)$ denotes the product of all numbers of the form $\left(1-\frac{1}{p}\right)$ with p taking as values the distinct prime divisors of n. ■

Coming back to our problem, let A, B be the sets of prime divisors of a, b respectively. Let us define the number α, β, γ in the following way

$$\alpha=\prod_{p\in A-B}\left(1-\frac{1}{p}\right), \qquad \beta=\prod_{p\in B-A}\left(1-\frac{1}{p}\right), \qquad \gamma=\prod_{p\in A\cap B}\left(1-\frac{1}{p}\right)$$

Now, by using (1) we have

$$\varphi(a)=a\alpha\gamma \quad , \quad \varphi(b)=b\beta\gamma \tag{2}$$

$$\varphi(a^2)=a^2\alpha\gamma \quad , \quad \varphi(b^2)=b^2\beta\gamma \quad , \quad \varphi(ab)=ab\alpha\beta\gamma \tag{3}$$

According to (2) and (3), we get

$$\begin{aligned}\frac{\varphi(ab)}{\sqrt{\varphi^2\left(a^2\right)+\varphi^2\left(b^2\right)}} &= \frac{ab\alpha\beta\gamma}{\sqrt{a^4\alpha^2\gamma^2+b^4\beta^2\gamma^2}} \leq \frac{ab\alpha\beta\gamma}{\sqrt{a^4\alpha^4\gamma^2+b^4\beta^4\gamma^2}} \leq \\ &\leq \frac{ab\alpha\beta\gamma}{\sqrt{2a^2b^2\alpha^2\beta^2\gamma^2}} = \frac{1}{\sqrt{2}}\end{aligned}$$

The latter majorization comes from the inequalities $\alpha, \beta \leq 1$ and

$$\left(a^2\alpha^2\gamma-b^2\beta^2\gamma\right)^2 \geq 0 \qquad \Leftrightarrow \qquad a^4\alpha^4\gamma^2+b^4\beta^4\gamma^2 \geq 2a^2b^2\alpha^2\beta^2\gamma^2 \qquad \square$$